

On the variational homotopy perturbation method for nonlinear oscillators

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In this paper we discuss a recent application of a variational homotopy perturbation method to rather simple nonlinear oscillators. We show that the main equations are inconsistent and for that reason the results may be of scarce utility.

I. INTRODUCTION

There has recently been great interest in developing simple solutions to textbook models of nonlinear oscillators[1–4] (and references therein). However, some of them are of questionable utility as shown, for example, by Rajendran et al[5] who concluded that He’s calculations of the limit cycle of the van der Pol oscillator[1] “contain several errors which once rectified make the method inapplicable to it”. I have disclosed several inconsistencies in a paper by Ren and He [2] and even proposed how to tidy up and improve their calculations [6].

Here I discuss a recent application of a variational homotopy perturbation method to rather simple nonlinear oscillators[4]. In Sec. II I analyse their results and in Sec. III draw conclusions.

II. VARIATIONAL HOMOTOPY PERTURBATION METHOD FOR NONLINEAR OSCILLATORS

Akbarzade and Langari[4] were interested in equations of the form

$$A(u) - f(r) = L(u) + N(u) - f(r) = 0 \quad (1)$$

where L and N are the linear and nonlinear parts of the operator A and u is the solution. They proposed the “homotopy perturbation structure”

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \quad (2)$$

where p is an embedding parameter (dummy perturbation parameter in the language of the well known perturbation theory) and u_0 is the first approximation that satisfies the boundary conditions.

They expanded the solution in p -power series $v = v_0 + v_1p + v_2p^2 + \dots$ and obtained the solution to Eq. (1) as $u = v_0 + v_1 + v_2 + \dots$ provided that the series converges for $p = 1$.

In particular, the authors concentrated in nonlinear oscillators of the form

$$u'' + \omega_0^2 u + \epsilon f(u) = 0 \quad (3)$$

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where f is a nonlinear function of u'' , u' and u , and considered the “variational functional” [4] (and references therein)

$$J(u) = \int_0^t \left[-\frac{1}{2}u'^2 + \frac{1}{2}\omega_0^2 u^2 + \epsilon F(u) \right] dt \quad (4)$$

where $dF/du = f$. Note that I have corrected a misprint in the authors’ Eq. (9). Obviously, $J(u)$ is minus the well known action integral[7] for a particular time interval.

In order to introduce the basic idea the authors first modified the well-known Duffing equation

$$u'' + u + \epsilon u^3 = 0, \quad u(0) = A, \quad u'(0) = 0 \quad (5)$$

as

$$u'' + \omega^2 u + p [\epsilon u^3 + (1 - \omega^2)u] = 0 \quad (6)$$

and derived the perturbation equations of order zero

$$u_0'' + \omega^2 u_0 = 0 \quad (7)$$

and first order

$$u_1'' + \omega^2 u_1 + \epsilon u_0^3 + (1 - \omega^2)u_0 = 0 \quad (8)$$

where

$$u_0(t) = A \cos(\omega t) \quad (9)$$

satisfies the boundary conditions and

$$u_1(0) = u_1'(0) = 0 \quad (10)$$

According to the authors “ ω will be identified from the variational formulation for u_1 , which reads”

$$J(u_1) = \int_0^T \left[-\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + (1 - \omega^2)u_0 u_1 + \epsilon u_0^3 u_1 \right] dt, \quad T = \frac{2\pi}{\omega} \quad (11)$$

They argued that the simplest trial function is[4]

$$u_1 = B \left[\cos(\omega t) - \frac{1}{3} \cos(5\omega t) \right] \quad (12)$$

Surprisingly, this function satisfies one of the boundary conditions $u_1'(0) = 0$ but not the other one because $u_1(0) = 2B/3 = 0$ leads to the unwanted trivial solution.

From the variational conditions $\partial J/\partial B = 0$ and $\partial J/\partial \omega = 0$ the authors obtained

$$\omega = \sqrt{1 + \frac{3}{4}\epsilon A^2}, \quad B = 0 \quad (13)$$

Although the estimated value of ω is reasonable, the result $B = 0$ leads to the trivial solution $u_1 \equiv 0$ that restricts considerably the practical utility of the approach.

In order to improve the results the authors proposed the correction

$$u_1 = B_1 \left(\cos(\omega t) - \frac{\cos(3\omega t)}{5} \right) + B_3 \left(\frac{\cos(3\omega t)}{5} - \frac{\cos(5\omega t)}{7} \right) \quad (14)$$

and from the variational conditions $\partial J/\partial B_1 = 0$, $\partial J/\partial B_3 = 0$, $\partial J/\partial \omega = 0$ they obtained the frequency

$$\omega = \frac{\sqrt{31}}{124} \sqrt{\sqrt{510237\rho^2 + 1416576\rho + 984064} - 357\rho - 496} \quad (15)$$

They did not show the coefficients; I obtained

$$B_1 = \frac{A [357\rho - 496 (\omega^2 - 1)]}{96\omega^2}, \quad B_3 = \frac{49A [3\rho - 4 (\omega^2 - 1)]}{96\omega^2} \quad (16)$$

Note that $u_1(t)$ does not satisfy one of the required boundary conditions

$$u_1(0) = -\frac{A (68\omega^2 - 49\rho - 68)}{16\omega^2} \quad (17)$$

and that the approximate solution $u_{app}(t) = u_0(t) + u_1(t)$ exhibits the wrong amplitude $u_{app}(0) = A + u_1(0)$. It therefore seems that by means of the variational approach the authors obtained a frequency for the amplitude A and an approximate trajectory $u_{app}(t)$ with a different amplitude.

Akbarzade and Langari[4] applied the method to other textbook nonlinear oscillators and obtained approximate frequencies. In all the cases they chose first-order corrections $u_1(t)$ that do not satisfy the boundary condition $u_1(0) = 0$ and obtained the trivial solution $u_1(t) \equiv 0$.

III. CONCLUSIONS

Although the combination of the homotopy perturbation method and the variational principle proposed by Akbarzade and Langari[4] led to reasonably approximate frequencies, one cannot take the approach seriously because of its inconsistencies. First, the first-order corrections to the solutions do not satisfy one of the boundary conditions proposed by the authors and, second, in most of the cases the resulting corrections are trivial (that is to say, they vanish identically). In the case where this correction does not vanish, the approximate solution exhibits an amplitude that is different from the one appearing in the expression for the frequency.

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